

OPTIMAL BOUNDS FOR THE NEUMAN-SÁNDOR MEANS IN TERMS OF GEOMETRIC AND CONTRA-HARMONIC MEANS

TIEHONG ZHAO, YUMING CHU, AND BAOYU LIU

ABSTRACT. In this article, we prove that the double inequality

$$\alpha G(a, b) + (1 - \alpha)C(a, b) < M(a, b) < \beta G(a, b) + (1 - \beta)C(a, b)$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 5/9$ and $\beta \leq 1 - 1/[2\log(1 + \sqrt{2})] = 0.4327 \dots$, where $G(a, b)$, $C(a, b)$ and $M(a, b)$ are respectively the geometric, contra-harmonic and Neuman-Sándor means of a and b .

1. INTRODUCTION

For $a, b > 0$ with $a \neq b$ the Neuman-Sándor mean $M(a, b)$ [1] is defined by

$$(1.1) \quad M(a, b) = \frac{a - b}{2 \sinh^{-1} \left(\frac{a-b}{a+b} \right)},$$

where $\sinh^{-1}(x) = \log(x + \sqrt{x^2 + 1})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean can be found in the literature [1, 2].

Let $G(a, b) = \sqrt{ab}$, $L(a, b) = (b - a)/(\log b - \log a)$, $P(a, b) = (a - b)/(4 \arctan \sqrt{a/b} - \pi)$, $A(a, b) = (a + b)/2$, $T(a, b) = (a - b)/[2 \arcsin((a - b)/(a + b))]$, $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ and $C(a, b) = (a^2 + b^2)/(a + b)$ be the geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic means of a and b , respectively. Then it is well-known that the inequalities

$$G(a, b) < L(a, b) < P(a, b) < A(a, b) < M(a, b) < T(a, b) < Q(a, b) < C(a, b)$$

hold true for $a, b > 0$ with $a \neq b$.

Neuman and Sándor [1, 2] established that

$$\begin{aligned} A(a, b) &< M(a, b) < T(a, b), \\ P(a, b)M(a, b) &< A^2(a, b), \\ A(a, b)T(a, b) &< M^2(a, b) < [A^2(a, b) + T^2(a, b)]/2 \end{aligned}$$

hold true for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1 - a$ and $b' = 1 - b$. Then the following Ky Fan inequalities

$$\frac{G(a, b)}{G(a', b')} < \frac{L(a, b)}{L(a', b')} < \frac{P(a, b)}{P(a', b')} < \frac{A(a, b)}{A(a', b')} < \frac{M(a, b)}{M(a', b')} < \frac{T(a, b)}{T(a', b')}$$

were presented in [1].

Let $L_p(a, b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ ($p \neq -1, 0$), $L_0 = 1/e(b^b/a^a)^{1/(b-a)}$ and $L_{-1}(a, b) = (b-a)/(\log b - \log a)$ be the p th generalized logarithmic mean of a and b . Li et al. [3] showed that the double inequality $L_{p_0}(a, b) < M(a, b) < L_2(a, b)$ holds true for all $a, b > 0$ with $a \neq b$, where $p_0 = 1.843 \dots$ is the unique solution of the equation $(p+1)^{1/p} = 2 \log(1 + \sqrt{2})$.

In [4], Neuman proved that the double inequalities

$$\alpha Q(a, b) + (1 - \alpha)A(a, b) < M(a, b) < \beta Q(a, b) + (1 - \beta)A(a, b)$$

and

$$\lambda Q(a, b) + (1 - \lambda)A(a, b) < M(a, b) < \mu Q(a, b) + (1 - \mu)A(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 - \log(\sqrt{2} + 1))/[(\sqrt{2} - 1) \log(\sqrt{2} + 1)] = 0.3249 \dots$, $\beta \geq 1/3$, $\lambda \leq (1 - \log(\sqrt{2} + 1))/\log(\sqrt{2} + 1) = 0.1345 \dots$ and $\mu \geq 1/6$.

1991 *Mathematics Subject Classification.* 26E60.

Key words and phrases. Neuman-Sándor mean, arithmetic mean, contra-harmonic mean.

This research was supported by the Natural Science Foundation of China under Grants 11071069 and 11171307, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

The main purpose of this paper is to find the least value α and the greatest value β such that the double inequality

$$\alpha G(a, b) + (1 - \alpha)C(a, b) < M(a, b) < \beta G(a, b) + (1 - \beta)C(a, b)$$

holds for all $a, b > 0$ with $a \neq b$.

Our main result is presented in Theorem 1.1.

Theorem 1.1. *The inequality*

$$(1.2) \quad \alpha G(a, b) + (1 - \alpha)C(a, b) < M(a, b) < \beta G(a, b) + (1 - \beta)C(a, b)$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 5/9$ and $\beta \leq 1 - 1/[2 \log(1 + \sqrt{2})] = 0.4327 \dots$.

2. LEMMAS

In order to prove our main result we need a lemma, which we present in this section.

Lemma 2.1. *Let $p \in (0, 1)$, $\lambda_0 = 1 - 1/[2 \log(1 + \sqrt{2})] = 0.4327 \dots$ and*

$$(2.1) \quad \varphi_p(t) = \sinh^{-1}(\sqrt{1 - t^2}) - \frac{\sqrt{1 - t^2}}{(p - 1)t^2 + pt + 2 - 2p}$$

Then $\varphi_{5/9}(t) < 0$ and $\varphi_{\lambda_0}(t) > 0$ for all $t \in (0, 1)$.

Proof. From (2.1), we have

$$(2.2) \quad \varphi_p(1) = 0,$$

$$(2.3) \quad \varphi_p(0) = \log(1 + \sqrt{2}) - \frac{1}{2(1 - p)},$$

$$(2.4) \quad \varphi'_p(t) = \frac{f_p(t)}{\sqrt{1 - t^2}\sqrt{2 - t^2}[(p - 1)t^2 + pt + 2 - 2p]^2} \quad t \in (0, 1)$$

where

$$(2.5) \quad f_p(t) = -t[(p - 1)t^2 + pt + 2 - 2p]^2 + [2(p - 1)t + p](1 - t^2)\sqrt{2 - t^2} + t\sqrt{2 - t^2}[(p - 1)t^2 + pt + 2 - 2p].$$

We divide the proof into two cases.

Case 1: $p = 5/9$. Then (2.5) leads to

$$(2.6) \quad f_{5/9}(1) = 0,$$

$$(2.7) \quad f'_{5/9}(t) = \frac{g_{5/9}(t)}{81\sqrt{2 - t^2}}$$

where

$$(2.8) \quad g_{5/9}(t) = -45t + 216t^2 - 144t^4 - (64 + 160t - 117t^2 - 160t^3 + 80t^4)\sqrt{2 - t^2}$$

We divide the discussion of this case into two subcases.

Subcase 1.1: $t \in (0, 3/4]$. Then from (2.8) we clearly see that

$$(2.9) \quad \frac{g_{5/9}(t)}{\sqrt{2 - t^2}} \leq 216t^2 - \frac{45t + 144t^4}{\sqrt{2}} - (64 + 160t - 117t^2 - 160t^3 + 80t^4) := \mu(t)$$

Differentiating $\mu(t)$ yields

$$(2.10) \quad \mu'(t) = -160 - \frac{45}{\sqrt{2}} + 666t + 480t^2 - (288\sqrt{2} + 320)t^3,$$

$$(2.11) \quad \mu''(t) = 666 + 960t - (960 + 864\sqrt{2})t^2.$$

From (2.10) and scientific computation we know that there exists unique $t_1 = 0.25869 \dots$ in $(0, 3/4]$ satisfying the equation $\mu'(t_1) = 0$. It follows from (2.11) that $\mu''(t_1) = 768.33 \dots > 0$ and t_1 is a unique extreme minimum point of $\mu(t)$ in $(0, 3/4]$. Therefore we obtain

$$(2.12) \quad \mu(t) \leq \max\{\mu(0), \mu(3/4)\} = -10.5824 \dots < 0.$$

Inequalities (2.9) and (2.12) lead to the conclusion that

$$g_{5/9}(t) < 0$$

for $t \in (0, 3/4]$.

Subcase 1.2: $t \in (3/4, 1)$. Then (2.8) gives

$$(2.13) \quad g'_{5/9}(t)\sqrt{2-t^2} = -320 + 532t + 1280t^2 - 991t^3 - 640t^4 + 400t^5 \\ - (45 - 432t + 576t^3)\sqrt{2-t^2} := h_{5/9}(t),$$

From (2.13) we get

$$(2.14) \quad h_{5/9}(1) = 72,$$

$$(2.15) \quad h'_{5/9}(t) = \frac{A(t)}{\sqrt{2-t^2}} + B(t)$$

where

$$(2.16) \quad A(t) = 864 + 45t - 4320t^2 + 2304t^4,$$

$$(2.17) \quad B(t) = 532 + 2560t - 2973t^2 - 2560t^3 + 2000t^4.$$

Equation (2.16) leads to

$$(2.18) \quad A'(3/4) = -2547 < 0,$$

$$(2.19) \quad A'(1) = 621 > 0,$$

$$(2.20) \quad A''(t) = -8640 + 27648t^2 > 0$$

for $3/4 < t < 1$.

From (2.18)-(2.20) we clearly see that there exists $t_2 \in (3/4, 1)$ such that $A(t)$ is strictly decreasing in $(3/4, t_2]$ and strictly increasing in $[t_2, 1)$. So we get

$$(2.21) \quad A(t) \leq \max\{A(3/4), A(1)\} = -803.25 \dots < 0.$$

Equation (2.15) and inequality (2.21) lead to

$$(2.22) \quad h'_{5/9}(t) < \frac{A(t)}{\sqrt{2}} + B(t) := \eta(t).$$

Computing $\mu(t)$ yields

$$(2.23) \quad \eta(3/4) = -235.484 \dots < 0,$$

$$(2.24) \quad \eta'(1) = -2626.886 \dots < 0,$$

$$(2.25) \quad \eta''(3/4) = 921.522 \dots > 0,$$

$$(2.26) \quad \eta'''(t) = 384[-40 + (125 + 72\sqrt{2})t] > \eta'''(3/4) = 49965.132 \dots > 0$$

for $t \in (3/4, 1)$.

Inequalities (2.23)-(2.26) implies that $\eta(t) < 0$ for $t \in (3/4, 1)$. From (2.8), (2.13), (2.14) and (2.22), we see that $g_{5/9}(t)$ is strictly increasing in $(3/4, 1)$. So we obtain

$$g_{5/9}(t) < g_{5/9}(1) = 0$$

for $3/4 < t < 1$.

Combining the last conclusions in subcases 1.1 and 1.2 we have

$$(2.27) \quad g_{5/9}(t) < 0$$

for all $t \in (0, 1)$.

Therefore, $\varphi_{5/9}(t) < 0$ for all $t \in (0, 1)$ follows easily from (2.2), (2.4), (2.6), (2.7) and (2.27).

Case 2: $p = \lambda_0 = 1 - 1/[2\log(1 + \sqrt{2})] = 0.4327 \dots$. Then (2.3) becomes

$$(2.28) \quad \varphi_{\lambda_0}(0) = 0$$

and (2.5) leads to

$$(2.29) \quad f_{\lambda_0}(0) = \sqrt{2}\lambda_0 > 0, \quad f_{\lambda_0}(1) = 0$$

$$(2.30) \quad f'_{\lambda_0}(t) = \frac{1}{4\delta^2} \left[\frac{C(t)}{\sqrt{2-t^2}} + D(t) \right]$$

where

$$\delta = \frac{1}{2(1-\lambda_0)} = \log(1 + \sqrt{2}) = 0.88137 \dots,$$

$$(2.31) \quad C(t) = 2\delta t(1 - 2\delta + 6t - 4t^3),$$

$$(2.32) \quad D(t) = -4 + (8 - 16\delta)t + (9 + 12\delta - 12\delta^2)t^2 + (16\delta - 8)t^3 - 5t^4.$$

We divide the discussion of this case into two subcases.

Subcase 2.1: $t \in (0, 1/2]$. Then differentiating (2.32) yields

$$(2.33) \quad D'(t) = 8 - 16\delta + (18 + 24\delta - 24\delta^2)t + (48\delta - 24)t^2 - 20t^3,$$

$$(2.34) \quad D''(t) = 6[3 + 4\delta - 4\delta^2 + (16\delta - 8)t - 10t^2].$$

From (2.33) and scientific computation we know that there exists unique $t_3 = 0.2555 \dots$ in $(0, 1/2]$ satisfying the equation $D'(t_3) = 0$. It follows from (2.34) that $D''(t_3) = 25.9469 \dots > 0$ and t_3 is a unique extreme minimum point of $D(t)$ in $(0, 1/2]$. So we have

$$(2.35) \quad D(t) \leq \max\{D(0), D(1/2)\} = -4 < 0$$

for $0 < t < 1/2$.

From (2.23) and (2.35) one has

$$(2.36) \quad 4\delta^2 \sqrt{2 - t^2} f'_{\lambda_0}(t) = C(t) + D(t) \sqrt{2 - t^2} < C(t) + D(t) := E(t).$$

It follows from (2.31) and (2.32) together with (2.36) that

$$(2.37) \quad E'(t) = 8 - 14\delta - 4\delta^2 + (18 + 48\delta - 24\delta^2)t + (48\delta - 24)t^2 - (20 + 32\delta)t^3,$$

$$(2.38) \quad E''(t) = 6[3 + 8\delta - 4\delta^2 + (16\delta - 8)t - (10 + 16\delta)t^2].$$

Computational and numerical experiments together with (2.37) show that there exists unique $t_4 = 0.17164 \dots$ in $(0, 1/2]$ satisfying the equation $E'(t_4) = 0$. It follows from (2.38) that $E''(t_4) = 43.686 \dots > 0$ and t_4 is a unique extreme minimum point of $E(t)$ in $(0, 1/2]$. Thus we obtain

$$(2.39) \quad E(t) \leq \max\{E(0), E(1/2)\} = -2.50591 \dots < 0$$

for $t \in (0, 1/2]$.

Inequalities (2.36) and (2.39) give

$$f'_{\lambda_0}(t) < 0$$

for $t \in (0, 1/2]$.

Subcase 2.2: $t \in (1/2, 1)$. Then (2.31) leads to

$$(2.40) \quad C'(t) = 2\delta(1 - 2\delta + 12t - 16t^3),$$

$$(2.41) \quad C''(t) = 24\delta(1 - 4t^2) < 0.$$

It follows from (2.40) and scientific computation that there exists unique $t_5 = 0.8322 \dots$ in $(1/2, 1)$ satisfying the equation $C'(t_5) = 0$. Then (2.41) leads to the conclusion that $C(t)$ is strictly increasing in $(1/2, t_5]$ and strictly decreasing in $(t_5, 1)$. Hence, we have

$$(2.42) \quad C(t) \geq \inf\{C(1/2), C(1)\} = 1.531 \dots > 0$$

for $t \in (1/2, 1)$.

On the other hand, equation (2.34) gives

$$(2.43) \quad D''(1/2) = 23.81 \dots > 0,$$

$$(2.44) \quad D''(1) = -2.87 \dots < 0,$$

$$(2.45) \quad D'''(t) = 24(4\delta - 2 - 5t) < 12(8\delta - 9) < 0$$

for $t \in (1/2, 1)$.

Equations (2.43)-(2.45) imply that there exists $t_6 \in (1/2, 1)$ such that $D'(t)$ is strictly increasing in $(1/2, t_6]$ and strictly decreasing in $[t_6, 1)$. Therefore, we have $D'(t) \geq \inf\{D'(1/2), D'(1)\} = 6.229 \dots > 0$ for $t \in (1/2, 1)$, which implies that $D(t)$ is strictly increasing in $(1/2, 1)$.

From the monotonicity of $D(t)$ in $(1/2, 1)$ and $C(t)$ in $(1/2, t_5]$ together with (2.42) we know that $C(t)/\sqrt{2 - t^2} + D(t)$ is strictly increasing in $(1/2, t_5]$. Equations (2.30)-(2.32) lead to

$$(2.46) \quad f'_{\lambda_0}(1/2) = -0.926 \dots < 0, \quad f'_{\lambda_0}(t_4) = 0.5193 \dots > 0.$$

It follows from (2.30) and (2.46) together with the monotonicity of $C(t)$ in $(1/2, t_5]$ that there exists $t_7 \in (1/2, t_5)$ such that $f'_{\lambda_0}(t) < 0$ for $t \in (1/2, t_7)$ and $f'_{\lambda_0}(t) > 0$ for $t \in (t_7, t_5]$.

If $t \in (t_5, 1)$, then from the monotonicity of $C(t)$ in $[t_5, 1)$ and $D(t)$ in $(1/2, 1)$ we have

$$(2.47) \quad \frac{C(t)}{\sqrt{2-t^2}} + D(t) > \frac{C(1)}{\sqrt{2}} + D(t_4) = 0.6859 \dots > 0$$

Equation (2.30) and (2.47) lead to $f'_{\lambda_0}(t) > 0$ for $t \in (t_5, 1)$. Therefore, we know that $f'_{\lambda_0}(t) < 0$ for $t \in (1/2, t_7)$ and $f'_{\lambda_0}(t) > 0$ for $t \in (t_7, 1)$.

Combining the last conclusions in subcases 2.1 and 2.2 we clearly see that $f_{\lambda_0}(t)$ is strictly decreasing in $(0, t_7]$ and strictly increasing in $[t_7, 1)$. Then (2.4) and (2.29) lead to the conclusion that there exists $t_0 \in (0, t_7)$ such that $\varphi_{\lambda_0}(t)$ is strictly increasing in $(0, t_0]$ and strictly decreasing in $[t_0, 1)$.

Therefore, $\varphi_{\lambda_0}(t) > 0$ for all $t \in (0, 1)$ follows from (2.2) and (2.28) together with the monotonicity of $\varphi_{\lambda_0}(t)$. □

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Since $M(a, b)$, $G(a, b)$ and $C(a, b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b$. Let $x = (a - b)/(a + b) \in (0, 1)$, $0 < p < 1$ and $\lambda_0 = 1 - 1/[2 \log(1 + \sqrt{2})] = 0.4327 \dots$. Then

$$(3.1) \quad \frac{C(a, b) - M(a, b)}{C(a, b) - G(a, b)} = \frac{(1 + x^2) \sinh^{-1}(x) - x}{(1 + x^2 - \sqrt{1 - x^2}) \sinh^{-1}(x)}$$

and

$$(3.2) \quad \begin{aligned} & pG(a, b) + (1 - p)C(a, b) - M(a, b) \\ &= A(a, b) \left[p\sqrt{1 - x^2} + (1 - p)(1 + x^2) - \frac{x}{\sinh^{-1}(x)} \right] \\ &= \frac{A(a, b)[p\sqrt{1 - x^2} + (1 - p)(1 + x^2)]}{\sinh^{-1}(x)} \varphi_p(\sqrt{1 - x^2}) \end{aligned}$$

where $\varphi_p(t)$ is defined as in Lemma 2.1.

Note that

$$(3.3) \quad \lim_{x \rightarrow 0^+} \frac{(1 + x^2) \sinh^{-1}(x) - x}{(1 + x^2 - \sqrt{1 - x^2}) \sinh^{-1}(x)} = \frac{5}{9},$$

$$(3.4) \quad \lim_{x \rightarrow 1^-} \frac{(1 + x^2) \sinh^{-1}(x) - x}{(1 + x^2 - \sqrt{1 - x^2}) \sinh^{-1}(x)} = 1 - \frac{1}{2 \log(1 + \sqrt{2})} = \lambda_0.$$

Equation (3.2) and Lemma 2.1 lead to the conclusion that the double inequality

$$\frac{5}{9}G(a, b) + \frac{4}{9}C(a, b) < M(a, b) < \lambda_0 G(a, b) + (1 - \lambda_0)C(a, b)$$

holds for all $a, b > 0$ with $a \neq b$.

- If $p_1 < 5/9$, then equations (3.1) and (3.3) imply that there exists $0 < \delta_1 < 1$ such that $M(a, b) < p_1 G(a, b) + (1 - p_1)C(a, b)$ for all $a, b > 0$ with $(a - b)/(a + b) \in (0, \delta_1)$.
- If $p_2 > \lambda_0$, then equations (3.1) and (3.4) imply that there exists $0 < \delta_2 < 1$ such that $M(a, b) > p_2 G(a, b) + (1 - p_2)C(a, b)$ for all $a, b > 0$ with $(a - b)/(a + b) \in (1 - \delta_2, 1)$.

Therefore, we conclude that in order for the inequalities (1.2) to be valid it is necessary and sufficient that $\alpha \geq 5/9$ and $\beta \leq 1 - 1/[2 \log(1 + \sqrt{2})] = 0.4327 \dots$. □

REFERENCES

- [1] E. NEUMAN and J. SÁNDOR, *On the Schwab-Borchardt mean*, Math. Pannon. **14**, 2(2003), 253-266.
- [2] E. NEUMAN and J. SÁNDOR, *On the Schwab-Borchardt mean II*, Math. Pannon **17**, 1(2006), 49-59.
- [3] Y.-M. LI, B.-Y. LONG and Y.-M. CHU, *Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean*, J. Math. Inequal. **6**, 4(2012), 567-577.
- [4] E. NEUMAN, *A note on a certain bivariate mean*, J. Math. Inequal. **6**, 4(2012), 637-643.

DEPARTMENT OF MATHEMATICS, HANGZHOU NORMAL UNIVERSITY, HANGZHOU 310036, P.R. CHINA
E-mail address: tiehongzhao@gmail.com

DEPARTMENT OF MATHEMATICS, HUZHOU TEACHERS COLLEGE, HUZHOU 313000, P.R. CHINA
E-mail address: chuyuming@hutc.zj.cn

SCHOOL OF SCIENCE, HANGZHOU DIANZI UNIVERSITY, HANGZHOU 310018, P.R. CHINA
E-mail address: 627847649@qq.com